Caringbah High School

Year 12 2024
Mathematics Extension 1
HSC Course
Assessment Task 4

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using black pen
- NESA-approved calculators may be used
- A reference sheet is provided
- In Questions 11-14, show relevant mathematical reasoning and/or calculations
- Marks may not be awarded for partial or incomplete answers

Total marks – 70

Section I **10 marks**

Attempt Questions 1-10

Mark your answers on the answer sheet provided. You may detach the sheet and write your name on it.

Section II **60 marks**

Attempt Questions 11-14

Write your answers in the answer booklets provided. Ensure your name or student number is clearly visible.

Name: ____ **Class:** ____________

Marker's Use Only						
Section I	**Section II**				**Total**	
Q 1-10	**Q11**	**Q12**	**Q13**	**Q14**		
/10	/15	/15	/15	/15	/70	%

Section I

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10

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1. What is the remainder when $x^3 - 8x$ is divided by $x + 4$?

   (A) 32

   (B) –32

   (C) $x^2 - 2x$

   (D) $x^2 - 4x + 8$

2. The solution to the inequality $\dfrac{3}{x-1} < 5$ can be expressed as:

   (A) $x \in \left[\dfrac{8}{5}, \infty\right)$

   (B) $x \in (-\infty, 1] \cup \left[\dfrac{8}{5}, \infty\right)$

   (C) $x \in \left(\dfrac{8}{5}, \infty\right)$

   (D) $x \in (-\infty, 1) \cup \left(\dfrac{8}{5}, \infty\right)$

3. A projectile has the equation of path $y = -3x^2 + 2x + 4$.

How far will it have travelled horizontally before it returns to its original position?

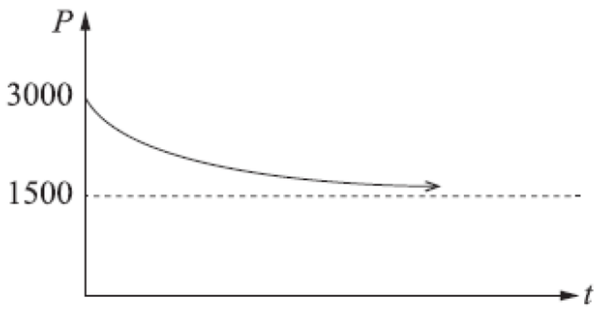
(A) $\frac{1}{3}$ units

(B) $\frac{2}{3}$ units

(C) 2 units

(D) 4 units

4. The diagram shows the number of penguins, $P(t)$, on an island at time t.



Which equation represents this graph?

(A) $P(t) = 1500 + 1500e^{-kt}$

(B) $P(t) = 3000 - 1500e^{-kt}$

(C) $P(t) = 3000 + 1500e^{-kt}$

(D) $P(t) = 4500 - 1500e^{-kt}$

5. What is the value of $\sin 2x$, given that $\sin x = \frac{2\sqrt{3}}{4}$ and x is obtuse?

(A) $-\frac{\sqrt{3}}{4}$

(B) $-\frac{\sqrt{3}}{2}$

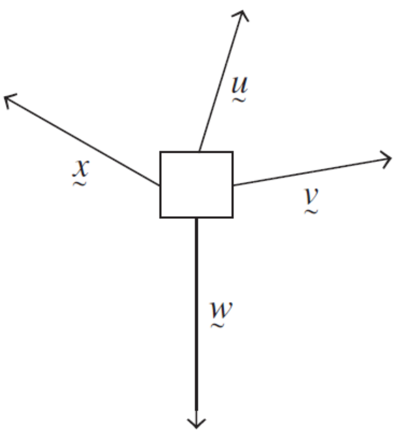
(C) $\frac{\sqrt{3}}{4}$

(D) $\frac{\sqrt{3}}{2}$

6. A box has four ropes attached to it. The ropes are being used to pull the box in four directions.

This is shown in the diagram, where:

- $\underset{\sim}{u} = \underset{\sim}{i} + 4\underset{\sim}{j}$
- $\underset{\sim}{v} = 3\underset{\sim}{i} + 2\underset{\sim}{j}$
- $\underset{\sim}{w} = a\underset{\sim}{i} - 7\underset{\sim}{j}$
- $\underset{\sim}{x} = -4\underset{\sim}{i} + b\underset{\sim}{j}$



If the box remains stationary while the ropes are being pulled, what are the values of a and b?

(A) $a = -\frac{4}{3}, b = -\frac{7}{8}$

(B) $a = 0, b = -1$

(C) $a = 0, b = 1$

(D) $a = 1, b = 1$

7. A florist is selecting flowers for a bouquet. He has four yellow, three red, five violet, seven white and five blue flowers to select from.
If the florist selects the flowers randomly, what is the minimum number of flowers that he would need to select to ensure that the bouquet has four flowers of the same colour?

(A) 15

(B) 16

(C) 19

(D) 20

8. What is the solution of the trigonometric equation $3\sin^2 x - 4\cos x + 1 = 0$ in the domain $0 \leq x \leq \pi$?

(A) $x = \dfrac{2}{3}$

(B) $x = \dfrac{2\pi}{3}$

(C) $x \approx 0.841$

(D) $x \approx 1.969$

9. What is the coefficient of x^3 in the binomial expansion $(2x - 3)^7$?

(A) -22680

(B) -15120

(C) 15120

(D) 22680

10. Which of the following is an expression for $\int \frac{3}{\sqrt{1-16x^2}}\,dx$?

(A) $3\sin^{-1}(4x) + C$

(B) $3\cos^{-1}(4x) + C$

(C) $\frac{3}{4}\sin^{-1}(4x) + C$

(D) $\frac{3}{4}\cos^{-1}(4x) + C$

End of Section I

Section II

60 marks
Attempt Questions 11–14
Allow about 1 hour 45 minutes for this section

Answer each question in a SEPARATE writing booklet.

Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Consider the vectors $\underset{\sim}{a} = 2\underset{\sim}{i} + 3\underset{\sim}{j}$ and $\underset{\sim}{b} = 9\underset{\sim}{i} + \underset{\sim}{j}$.

(i) Calculate the angle, in radians to 3 decimal places, between the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$. **2**

(ii) Find the vector projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$. **2**

(b) **(i)** Express $\cos x - \sqrt{3}\sin x$ in the form $R\cos(x + \alpha)$ where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. **2**

(ii) Hence, solve $\cos x - \sqrt{3}\sin x = -2$ for $0 \leq x \leq 2\pi$ **2**

(c) Evaluate $\int_0^{\frac{\pi}{4}} \sin^2 x\,dx$ **2**

(d) When $P(x)$ is divided by $(x + 1)$ the remainder is 3, and when $P(x)$ is divided by $(x - 2)$ the remainder is -5. **2**

What is the remainder when $P(x)$ is divided by $(x + 1)(x - 2)$?

Question 11 continues on page 8

(e) **(i)** Find the largest possible domain of the function $f(x) = x^2 - 4x + 5$ for which $f(x)$ has an inverse function $f^{-1}(x)$. **1**

(ii) Find $f^{-1}(x)$ and hence sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes. **2**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) Use induction to prove that $4^n + 14$ is divisible by 6 for all positive integers n. **3**

(b) Use the substitution $u = \tan x$ to evaluate $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \tan^3 x \sec^2 x\, dx$ **3**

(c) Given $\underset{\sim}{a} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$

(i) Find $\underset{\sim}{a} + \underset{\sim}{b}$ and $\underset{\sim}{a} - \underset{\sim}{b}$ **2**

(ii) Hence, find $(\underset{\sim}{a} + \underset{\sim}{b}) \cdot (\underset{\sim}{a} - \underset{\sim}{b})$. State if $\underset{\sim}{a} + \underset{\sim}{b}$ is perpendicular to $\underset{\sim}{a} - \underset{\sim}{b}$ and justify your answer with a reason. **2**

(d) The letters of the word CIRCLE are written at random on the circumference of a circle.

(i) How many different arrangements are possible? **1**

(ii) What is the probability that the C's are separated? **2**

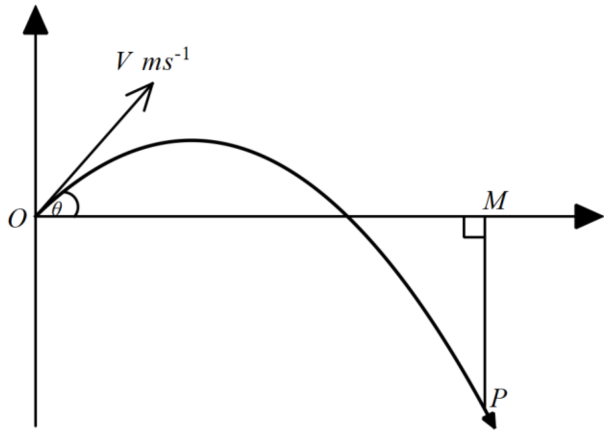
(e) Find the values of a and b for which **2**

$$(ax + b)^7 \equiv 128x^7 - 2\,240x^6 + \ldots$$

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a)



A projectile is fired from a point O with initial speed V ms^{-1} at an angle of elevation of θ. It is subject to a force which gives it a constant, vertically downward acceleration of magnitude g ms^{-2}.

(i) Find functions for its horizontal (x) and vertical (y) displacements from O after t seconds. **2**

(ii) The projectile falls to a point P, below the level of O, such that $PM = OM$. **2**

Prove that the time taken to reach P is $\dfrac{2V(\sin\theta + \cos\theta)}{g}$ seconds.

(iii) Show that the distance OM is $\dfrac{V^2(\sin 2\theta + \cos 2\theta + 1)}{g}$ metres. **2**

(iv) If the horizontal range of the projectile level with O is r m and $OM = \dfrac{4r}{3}$, prove that $\sin 2\theta - 3\cos 2\theta = 3$. **2**

Question 13 continues on page 11

(b) **(i)** Prove that $\dfrac{d}{dx}(\sec x) = \sec x \tan x$ **1**

(ii) If $y = \ln(\sec x + \tan x)$, find $\dfrac{dy}{dx}$ in simplest form. **2**

(iii) Evaluate $\displaystyle\int_0^1 \sec x \, dx$ correct to 2 decimal places. **2**

(c) Simplify $2\sin^2\theta\cos\theta\cos\varphi + 2\sin\theta\cos^2\theta\sin\varphi$ **2**

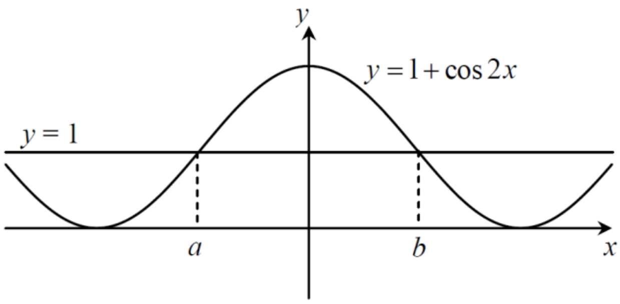
End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) Use trigonometric identities to prove that **2**

$$\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \frac{1}{\cos\left(\frac{\pi}{2}-\theta\right)} - \cot\theta.$$

(b)



Part of the curve $y = 1 + \cos 2x$ is shown above.

The curve cuts the line $y = 1$ at $x = a$ and $x = b$.

(i) Write down the values of a and b. **1**

(ii) The area between the curve $y = 1 + \cos 2x$ and $y = 1$ from $x = a$ to $x = b$ is rotated about the x-axis. **3**

Find the exact volume of the solid generated.

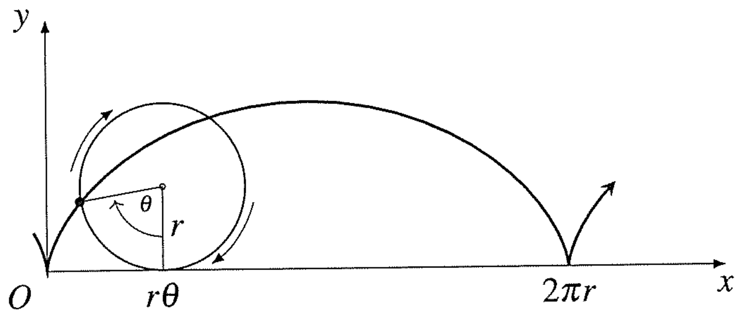
(c) The interior surface of a bowl is made by rotating the curve $y = \dfrac{x^2}{4}$ about the y-axis, the units on both axes being in centimetres (cm). Water is poured into the bowl at a constant rate of 36cm^3 per second. When the depth of the water is h cm, the area of the surface of the water is S cm^2. Find:

(i) the rate of increase in h when $S = 12$. **2**

(ii) the rate of increase in S when $h = 9$. **2**

Question 14 continues on page 13

(d) A cycloid is the path traced by a point on the circumference of a circle as it rolls along the x-axis. **5**



For a circle of radius r, it is described by the parametric equations

$$x = r(\theta - \sin\theta)$$
$$y = r(1 - \cos\theta)$$

Where θ is the angle the radius to the particular point on the circumference makes with a vertical line.

The arc length l can be calculated by evaluating

$$l = \int_{\theta_2}^{\theta_1} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta.$$

Find the distance, in terms of r, a point on the circumference travels in one complete rotation.

End of Examination

Multiple choice

① $P(-4) = (-4)^3 - 8(-4)$ — (B)

$= -32$

② $\dfrac{3}{x-1} < 5$ $\qquad x \neq 1$

$3(x-1) = 5(x-1)^2$

$3x - 3 = 5x^2 - 10x + 5$

$5x^2 - 13x + 8 = 0$

$5x^2 - 5x - 8x + 8 = 0$

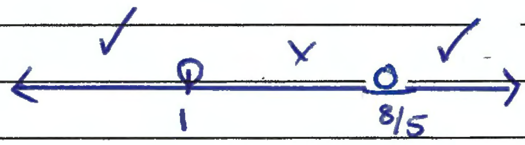
$5x(x-1) - 8(x-1) = 0$

$(x-1)(5x-8) = 0$

$\therefore x = 1 \qquad 5x - 8 = 0$

$5x = 8$

$x = \dfrac{8}{5}$



$\therefore x \in (-\infty, 1) \cup (8/5, \infty)$ — (D)

③ $x = 0 \;,\; y = 4$

$\therefore 4 = -3x^2 + 2x + 4$

$3x^2 - 2x = 0$

$x(3x-2) = 0$

$x = 0 \;,\quad 3x - 2 = 0$ — (B)

$x = \dfrac{2}{3}$

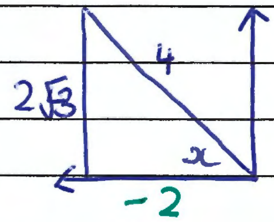
(4) when $t=0$ $P(0) = 3000$ $\therefore$ only B or C

as t increases $P(t)$ decreases (B)

$\therefore$ 3000 minus.

(5) $\sin 2x = 2\sin x \cos x$

x is obtuse



$\therefore \cos x = -\frac{1}{2}$

$\sin 2x = 2 \times \frac{2\sqrt{3}}{4} \times -\frac{1}{2}$ (B)

$= -\frac{\sqrt{3}}{2}$

(6) $1+3+a-4 = 0$ $\quad 4+2-7+b=0$ (C)

$a=0$ $\quad b=1$

(7) 4 yellow
3 red $\quad$ 3 of each +1
5 violet
7 white $\quad = 3\times5+1$ (B)
5 blue $\quad = 16$

(8) $3(1-\cos^2 x) - 4\cos x + 1 = 0$

$3 - 3\cos^2 x - 4\cos x + 1 = 0$

$3\cos^2 x + 4\cos x - 4 = 0$

$\cos x = \frac{-4 \pm \sqrt{16 - 4\times3\times-4}}{6}$

$\cos x = \frac{-4 \pm 8}{6}$

$= 2/3,\ \cancel{-2}$

$\cos x = 2/3$ (C)

$x = 0.8410\ldots\ldots$

9 $(2x-3)^7$ x^3 term.

${}^7C_3(2x)^3 \times (-3)^4 = 22680x^3$ (D)

10 $\int \frac{3}{\sqrt{1-16x^2}}\,dx$

$= 3\int \frac{1}{\sqrt{1-16x^2}}\,dx$

$= \frac{3}{4}\int \frac{4}{\sqrt{1-16x^2}}\,dx$ (C)

$= \frac{3}{4}\sin^{-1}(4x) + C$

Question 11

a) $\underset{\sim}{a} = 2\underset{\sim}{i} + 3\underset{\sim}{j}$ $\quad\quad$ $\underset{\sim}{b} = 9\underset{\sim}{i} + \underset{\sim}{j}$

i) $\underset{\sim}{a} \cdot \underset{\sim}{b} = (2 \times 9) + (3 \times 1)$
$= 21$

$|\underset{\sim}{a}| = \sqrt{4 + 9} = \sqrt{13}$

$|\underset{\sim}{b}| = \sqrt{81 + 1} = \sqrt{82}$

$\therefore \theta = \cos^{-1} \dfrac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}| \cdot |\underset{\sim}{b}|}$

$= \cos^{-1} \dfrac{21}{\sqrt{13} \cdot \sqrt{82}}$

$= 0.872136 \ldots$
$\doteqdot 0.872 \ (3dp)$

ii) $\text{proj}_{\underset{\sim}{b}} \underset{\sim}{a} = \dfrac{\underset{\sim}{b} \cdot \underset{\sim}{a}}{|\underset{\sim}{b}|^2} \cdot \underset{\sim}{b}$

$= \dfrac{21}{(\sqrt{82})^2} \cdot (9\underset{\sim}{i} + \underset{\sim}{j})$

$= \dfrac{189}{82}\underset{\sim}{i} + \dfrac{21}{82}\underset{\sim}{j}$

b) i) $\cos x - \sqrt{3}\sin x \Rightarrow R\cos(x + a)$

$r = \sqrt{1^2 + (\sqrt{3})^2}$
$= \sqrt{4}$
$= 2$

$\tan\alpha = \dfrac{\sqrt{3}}{1}$

$\alpha = \dfrac{\pi}{3}$

$\therefore \cos x - \sqrt{3}\sin x = 2\cos\left(x + \dfrac{\pi}{3}\right)$

ii) $\cos x - \sqrt{3}\sin x = -2 \qquad 0 \le x \le 2\pi$

$\therefore \quad 2\cos\left(x + \frac{\pi}{3}\right) = -2 \qquad -\frac{\pi}{3} \le x + \frac{\pi}{3} \le \frac{7\pi}{3}$

$\cos\left(x + \frac{\pi}{3}\right) = -1$

$x + \frac{\pi}{3} = \pi$

$x = \frac{2\pi}{3}$

c) $\int_0^{\pi/4} \sin^2 x \, dx$

$= \frac{1}{2}\int_0^{\pi/4} (1 - \cos 2x) \, dx$

$= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\pi/4}$

$= \frac{1}{2}\left[\left(\frac{\pi}{4} - \frac{1}{2}\sin\frac{\pi}{2}\right) - \left(0 - \frac{1}{2}\sin 0\right)\right]$

$= \frac{\pi}{8} - \frac{1}{4}$

d) $P(x) = Q(x) \cdot D(x) + R(x)$

$P(-1) = Q(x) \cdot (x+1) + 3$

$P(2) = Q(x) \cdot (x-2) - 5$

$\therefore \quad P(x) = (x+1)(x+2) \cdot Q(x) + (ax + b)$

$P(-1) = 0 + (-a + b)$

$\therefore \quad -a + b = 3 \quad (1)$

$P(2) = 0 + 2a + b$

$\therefore \quad 2a + b = -5 \qquad (2)$

$(2) - (1) \quad 3a = -8$

$\boxed{a = -8/3}$

sub into (1) $\quad 8/3 + b = 3$

$\boxed{b = 1/3}$

$\therefore$ Remainder $= -\frac{8x}{3} + \frac{1}{3}$

$= -\frac{1}{3}(8x - 1)$

e) i) $f(x) = x^2 - 4x + 5$

$f'(x) = 2x - 4$

$2x - 4 = 0$
$2x = 4$
$x = 2$

$\therefore \ (-\infty, 2]$ $\underline{\text{OR}}$ $[2, \infty)$ are possible domains
(Only need 1)

ii) $x = y^2 - 4y + 5$

$x = y^2 - 4y + 4 + 1$

$x = (y-2)^2 + 1$

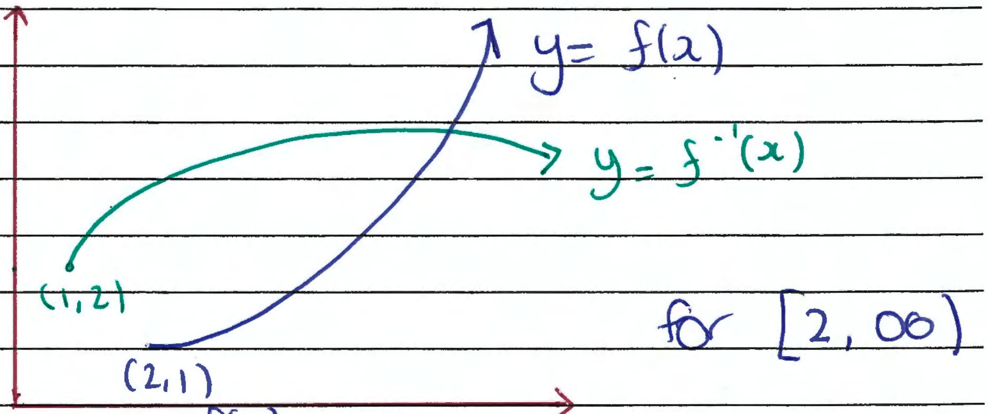
$x - 1 = (y-2)^2$

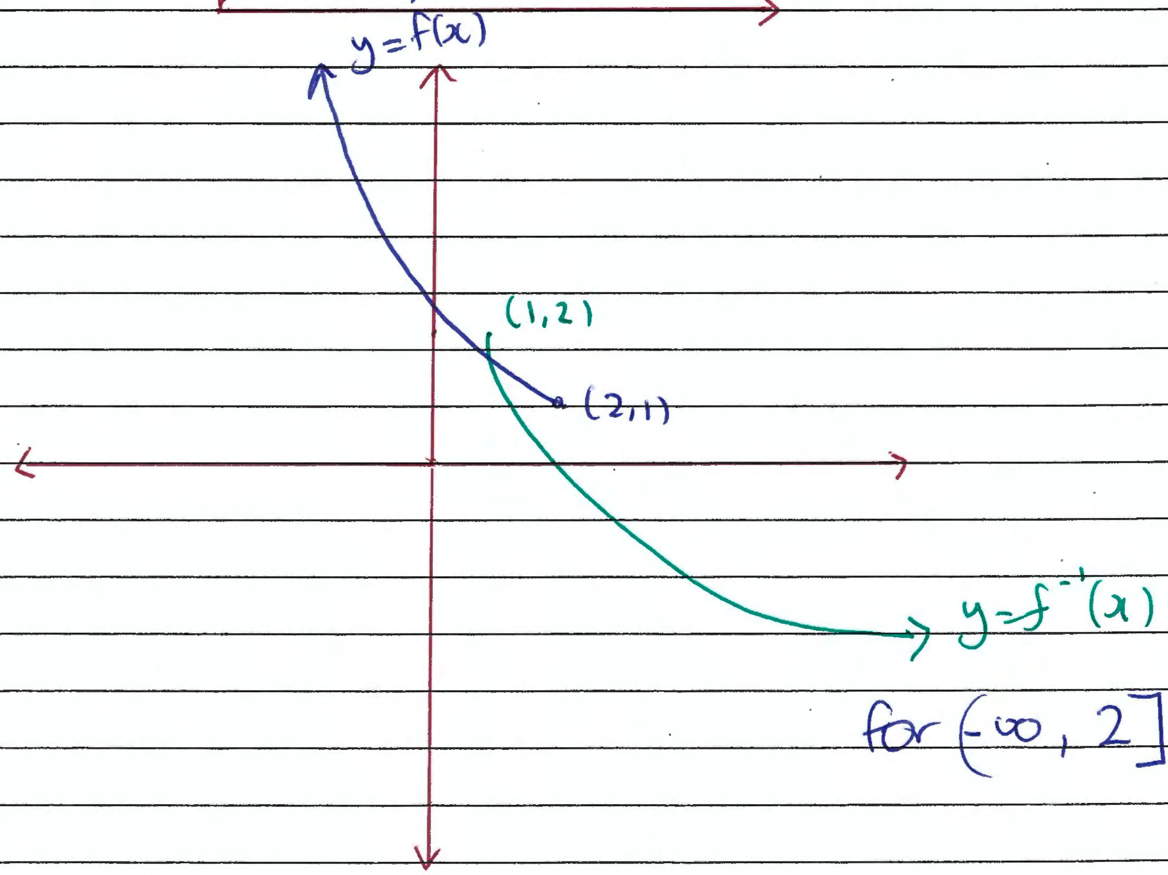
$$y - 2 = \pm\sqrt{x-1}$$

$$y = 2 \pm \sqrt{x-1}$$

$\therefore f^{-1}(x) = 2 + \sqrt{x-1}$ <u>OR</u> $2 - \sqrt{x-1}$

answer must correlate to Ss answer from e i)





Question 12

a) prove true for $n=1$

$4^1+14=18 \quad \therefore \div$ by 6

assume true for $n=k$

let $4^k+14=6M$ for some integer M

$\therefore 4^k = 6M-14$

prove true for $n=k+1$

$$4^{k+1}+14 = (4^k \times 4^1)+14$$

$$= (6M-14)\times 4^1 + 14 \quad \text{from assumption}$$

$$= 4\times 6M - 56 + 14$$

$$= 4\times 6M - 42$$

$$= 6(4m-7)$$

$$= 6P, \text{ where } 4m-7 = P$$

$\therefore 4^{k+1}+14$ is divisible by 6.

Since the statement is true for $n=1$ & is true for $n=k+1$, by the process of mathematical induction the statement is true for all values of n.

b) $\int_{\pi/4}^{\pi/3} \tan^3 x \sec^2 x \, dx$

$u = \tan x$

$\frac{du}{dx} = \sec^2 x$

$du = \sec^2 x \, dx$

$= \int_{1}^{\sqrt{3}} u^3 \, du$

$u = \tan \pi/3$
$= \sqrt{3}$

$= \left[\frac{u^4}{4} \right]_1^{\sqrt{3}}$

$u = \tan \pi/4$
$= 1$

$= \frac{9}{4} - \frac{1}{4}$

$= 2 \text{ units}^2$

c) i) $\underset{\sim}{a} + \underset{\sim}{b} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 \\ 7 \end{pmatrix}$

$= \begin{pmatrix} 7 \\ 9 \end{pmatrix}$

$\underset{\sim}{a} - \underset{\sim}{b} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 4 \\ 7 \end{pmatrix}$

$= \begin{pmatrix} -1 \\ -5 \end{pmatrix}$

ii) $(\underset{\sim}{a} + \underset{\sim}{b}) \cdot (\underset{\sim}{a} - \underset{\sim}{b}) = (7 \times -1) + (9 \times -5)$

$= -52$

Since $(\underset{\sim}{a} + \underset{\sim}{b}) \cdot (\underset{\sim}{a} - \underset{\sim}{b}) \neq 0$ $\underset{\sim}{a} + \underset{\sim}{b}$ is <u>not</u> perpendicular to $\underset{\sim}{a} - \underset{\sim}{b}$

d) i) C I R C L E

$$\frac{(n-1)!}{2!} = \frac{5!}{2!}$$

$$= 60$$

ii) P(C's separated) = 1 − P(C's together)

$$= 1 - \frac{\frac{4!}{2!} \times 2!}{60}$$

$$= \frac{3}{5}$$

e) $(ax+b)^7 \equiv 128x^7 - 2240x^6 + \ldots\ldots$

$${}^7C_0 \times (ax)^7 \times b^0 = 128x^7$$

$$a^7x^7 = 128x^7$$

$$\therefore a^7 = 128$$

$$\boxed{a = 2}$$

$${}^7C_1 \times (ax)^6 \times b = -2240x^6$$

$$7a^6bx^6 = -2240x^6$$

$$\therefore a^6b = -320$$

$$2^6b = -320$$

$$\boxed{b = -5}$$

Question 13

a) i) $\ddot{x} = 0$ $\ddot{y} = -g$

$\dot{x} = V\cos\theta$ $\dot{y} = -gt + V\sin\theta$

$x = Vt\cos\theta$ $y = -\frac{gt^2}{2} + Vt\sin\theta$

ii) $PM = OM$ $\therefore$ $x = -y$

$\therefore$ $Vt\cos\theta = -\left(-\frac{gt^2}{2} + Vt\sin\theta\right)$

$= \frac{gt^2}{2} - Vt\sin\theta$

$\frac{gt^2}{2} = Vt\cos\theta + Vt\sin\theta$

$t^2 = 2\left(\frac{Vt\cos\theta + Vt\sin\theta}{g}\right)$

$t = \frac{2V(\sin\theta + \cos\theta)}{g}$

iii) $OM = x$ when $t = \uparrow$

$\therefore$ $x = V \cdot \frac{2V(\sin\theta + \cos\theta)}{g} \cdot \cos\theta$

$= \frac{2V^2(\cos^2\theta + \cos\theta\sin\theta)}{g}$

$= \frac{V^2(2\cos^2\theta + 2\cos\theta\sin\theta)}{g}$

$= \frac{V^2(1 + \cos 2\theta + \sin 2\theta)}{g}$

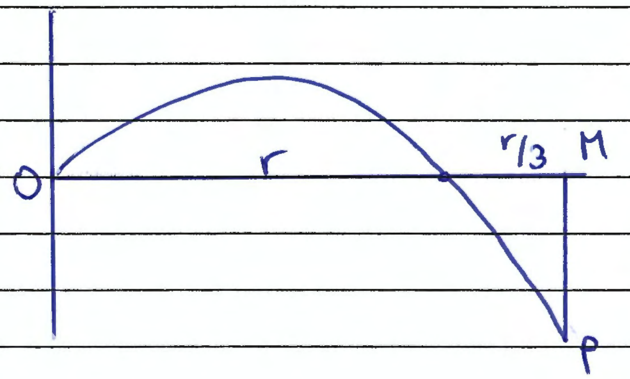
iv) when $y=0$ $\qquad 0 = -\frac{gt^2}{2} + vt\sin\theta$

$$0 = t\left(-\frac{gt}{2} + v\sin\theta\right)$$

$$0 = -\frac{gt}{2} + v\sin\theta$$

$$\frac{gt}{2} = v\sin\theta$$

$$\therefore t = \frac{2v\sin\theta}{g}$$



$r = vt\cos\theta$ $\qquad$ but $\qquad t = \frac{2v\sin\theta}{g}$

$$\therefore r = \frac{2v^2\sin\theta\cos\theta}{g}$$

$$OM = \frac{4r}{3}$$

$$\therefore \frac{4}{3} \times \left(\frac{2\cancel{v^2}\sin\theta\cos\theta}{\cancel{g}}\right) = \frac{\cancel{v^2}(1 + \cos 2\theta + \sin 2\theta)}{\cancel{g}}$$

$$\frac{4}{3}\sin 2\theta = 1 + \cos 2\theta + \sin 2\theta$$

$$4\sin 2\theta = 3 + 3\cos 2\theta + 3\sin 2\theta$$

$$\therefore \sin 2\theta - 3\cos 2\theta = 3$$

b) i) $\frac{d}{dx} \sec x = \frac{d}{dx} \frac{1}{\cos x}$

$= \frac{\cos x \times 0 - 1 \times \sin x}{\cos^2 x}$

$= \frac{\sin x}{\cos^2 x}$

$= \frac{\sin x}{\cos x \cos x}$

$= \frac{\tan x}{\cos x}$

$= \sec x \tan x$

ii) $y = \ln(\sec x + \tan x)$

$\frac{dy}{dx} = \frac{f'(x)}{f(x)}$

$= \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}$

$= \frac{\sec x(\tan x + \sec x)}{\sec x + \tan x}$

$= \sec x$

iii) $\int_0^1 \sec x \, dx = \left[\ln(\sec x + \tan x)\right]_0^1$

$= [\ln(\sec 1 + \tan 1) - \ln(\sec 0 + \tan 0)]$

$= 0.01745 \ldots$

$= 0.02$ (2dp)

c) $2\sin^2\theta\cos\theta\cos\phi + 2\sin\theta\cos^2\theta\sin\phi$

$= \sin 2\theta \sin\theta\cos\phi + \sin 2\theta\cos\theta\sin\phi$

$= \sin 2\theta(\sin\theta\cos\phi + \cos\theta\sin\phi)$

$= \sin 2\theta(\sin(\theta+\phi))$

Question 14

a) $LHS = \sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}}$

$= \sqrt{\dfrac{(1-\cos\theta)(1-\cos\theta)}{(1+\cos\theta)(1-\cos\theta)}}$

$= \sqrt{\dfrac{(1-\cos\theta)^2}{1-\cos^2\theta}}$

$= \sqrt{\dfrac{(1-\cos\theta)^2}{\sin^2\theta}}$

$= \dfrac{1-\cos\theta}{\sin\theta}$

$= \dfrac{1}{\sin\theta} - \dfrac{\cos\theta}{\sin\theta}$

$= \dfrac{1}{\cos\left(\frac{\pi}{2}-\theta\right)} - \cot\theta$

$= RHS$

b) i) $1 + \cos 2x = 1$

$\cos 2x = 0$

$2x = \pm\dfrac{\pi}{2}$

$x = \pm\dfrac{\pi}{4}$

$\therefore\ a = -\dfrac{\pi}{4},\quad b = \dfrac{\pi}{4}$

ii) $V = \pi \left[\int_{-\pi/4}^{\pi/4} (1 + \cos 2x)^2 \, dx - \int_{-\pi/4}^{\pi/4} 1 \, dx \right]$

$= 2\pi \left[\int_0^{\pi/4} (1 + 2\cos 2x + \cos^2 2x - 1) \, dx \right]$

$= 2\pi \left[\int_0^{\pi/4} (2\cos 2x + \cos^2 2x) \, dx \right]$

$= 2\pi \left[\int_0^{\pi/4} \left(2\cos 2x + \frac{1}{2}(1 + \cos 4x)\right) dx \right]$

$= 2\pi \left[\sin 2x + \frac{1}{2}\left(x + \frac{1}{4}\sin 4x\right) \right]_0^{\pi/4}$

$= 2\pi \left[\left(\sin \frac{\pi}{2} + \frac{1}{2}\left(\frac{\pi}{4} + \frac{1}{4}\sin \pi\right)\right) - \left(\sin 0 + \frac{1}{2}\left(0 + \frac{1}{4}\sin 0\right)\right) \right]$

$= 2\pi \left(1 + \frac{1}{2}\left(\frac{\pi}{4} + 0\right) - 0 \right)$

$= 2\pi + \frac{\pi^2}{4} \quad u^3$

c) i) $y = \frac{x^2}{4} \Rightarrow$ about the y axis

$\therefore x^2 = 4y$

$\frac{dV}{dt} = 36 \text{ cm}^3/\text{s}$

when $y = h$ $\quad SA = S \text{ cm}^2$

need to find $\frac{dh}{dt}$ $\therefore$ need $\frac{dh}{dV}$

Find V by integrating $x^2 = 4y$

$$V = \pi \int_0^h 4y \, dy$$

$$= \pi \left[\frac{4y^2}{2} \right]_0^h$$

$$= 2\pi h^2$$

$$\frac{dV}{dh} = 4\pi h$$

when $y = h$, $S = 12$

$\therefore \; 12 = \pi r^2 \quad (r = x)$

$\therefore \; 12 = \pi x^2$

$\therefore x^2 = \frac{12}{\pi}$

but $x^2 = 4y$

$\therefore 4y = \frac{12}{\pi}$

$y = \frac{3}{\pi}$

$\therefore \boxed{h = \frac{3}{\pi}}$

$$\therefore \frac{dV}{dh} = 4\pi \times \frac{3}{\pi}$$

$$= 12$$

$$\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV}$$

$$= 36 \times \frac{1}{12}$$

$$= 3 \text{ cm/s}$$

ii) find $\frac{dS}{dt}$ when $h = 9$

$\therefore \; 9 = \frac{x^2}{4}$

$x^2 = 36$

$x = 6$

$S = \pi x^2$

$\therefore S = 4\pi h$

$\frac{dS}{dh} = 4\pi$ $\qquad$ $\frac{dS}{dt} = \frac{dh}{dt} \times \frac{dS}{dh}$

$\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV}$ from i)

$= 36 \times \frac{1}{4\pi h}$

$= \frac{9}{\pi h}$

when $h = 9$ $\quad \frac{dh}{dt} = \frac{1}{\pi}$

$\therefore \frac{dS}{dt} = \frac{1}{\pi} \times 4\pi$

$\boxed{\frac{dS}{dt} = 4 \text{ cm}^2/\text{s}}$

d) $x = r(\theta - \sin\theta)$ $\qquad$ $y = r(1 - \cos\theta)$

$\frac{dx}{d\theta} = r(1 - \cos\theta)$ $\qquad$ $\frac{dy}{d\theta} = r\sin\theta$

$\therefore l = \int_0^{2\pi} \sqrt{(r^2(1-\cos\theta)^2 + r^2\sin^2\theta)}\, d\theta$

$= r\int_0^{2\pi} \sqrt{(1 - 2\cos\theta + \cos^2\theta + 1 - \cos^2\theta)}\, d\theta$

$$= r\int_0^{2\pi} \sqrt{2 - 2\cos\theta} \quad d\theta$$

$$= r\int_0^{2\pi} \sqrt{2(1 - \cos\theta)} \quad d\theta$$

$$= r\sqrt{2} \int_0^{2\pi} \sqrt{\left(2\sin^2\frac{\theta}{2}\right)} \, d\theta$$

$$= 2r\int_0^{2\pi} \sin\frac{\theta}{2} \quad d\theta$$

$$= \frac{2r}{1/2} \int_0^{2\pi} \frac{1}{2}\sin\frac{\theta}{2} \quad d\theta$$

$$= 4r\left[-\cos\frac{\theta}{2}\right]_0^{2\pi}$$

$$= 4r\left[-\cos\theta \quad + \cos 0\right]$$

$$= 4r\left[1 \quad + 1\right]$$

$$= 8r$$